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Efficient Algorithms for Solving Certain Nonconvex Programs Dealing with the Product of Two Affine Fractional Functions

LE D. MUU and BUI T. TAM
Institute of Mathematics, Hanoi, Vietnam

and

S. SCHAIBLE
Graduate School of Management, University of California, Riverside, CA 92521, U.S.A.

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Abstract. Two algorithms for finding a global minimum of the product of two affine fractional functions over a compact convex set and solving linear fractional programs with an additional constraint defined by the product of two affine fractional functions are proposed. The algorithms are based on branch and bound techniques using an adaptive branching operation which takes place in one-dimensional intervals. Results from numerical experiments show that large scale problems can be efficiently solved by the proposed methods.

Key words: Product of two fractional functions, global optimization, branch and bound, adaptive branching, efficiency.

1. Introduction

Branch and bound techniques are the most commonly used for the solution of nonconvex global optimization problems. Branch and bound methods differ in the way they define rules for branching and the methods used for deriving bounds. They avoid exhaustive research.

In their recent paper [8] Muu and Oettli developed a branch and bound algorithm for minimizing a convex – concave function over a convex set. There the branching operation is an adaptive bisection whereas the bounding operation is based on a suitable relaxation of the objective function. The main disadvantage of these operations is that they require searching the vertices of the polyhedral convex sets encountered in the algorithm. The number of these vertices may increase exponentially with the number of iterations. Determining them is the most costly part of the algorithm.

In this paper we extend the algorithm proposed in [8] for minimizing the product of two affine fractional functions over a convex set, and for solving linear fractional programs with an additional constraint defined by the product of two affine fractional functions. We show that for these problems all polyhedral convex sets encountered in the algorithm are one-dimensional intervals (hence having exactly two vertices).

The product of two affine fractional functions appears in the bond portfolio optimization model [4]. A special case is the product of two affine functions which has some applications in VLSI chip design, microeconomics and transportation. The problems dealing with the product of two affine functions recently have been considered by a number of authors (see, e.g. [2, 5, 6, 9, 10, 11, 13, 14]). In [5] Konno and Yajima developed two algorithms for minimizing and maximizing the product of two affine fractional functions over a polytope. Their first algorithm is a parametric simplex method with two parameters. The second algorithm is a branch and bound one in which for calculating lower bounds it requires minimizing the sum of two affine fractional functions. As it is reported in their paper, the second algorithm is more efficient than the first one.

In the algorithms that we shall describe in the next sections, the bounding operation requires solving only linear fractional subprograms. In contrast to the methods in [5], we do not have to partition the feasible region according to the sign of the affine fractional functions involved in the problems. Results from numerical experiments show that large scale problems can be efficiently solved by the proposed method.

2. An Algorithm for Minimizing the Product of Two Affine Fractional Functions with Convex Constraints

We consider first the following problem, denoted by (P):

$$\min\left\{f(x) := \frac{f_1(x)}{f_2(x)}\frac{f_3(x)}{f_4(x)} : x \in D\right\}, \tag{P}$$

where f_i $(i = 1, \ldots, 4)$ are affine functions defined by $f_i(x) = c_i^T x + b_i$ and D is a compact convex set in R^n. We assume throughout the paper that $f_i(x) > 0$ $(i = 2, 4)$ for all $x \in D$. Hence the objective function $f(x) := (f_1(x)f_3(x))/(f_2(x)f_4(x))$ is continuous on D.

Take $t := f_1(x)/f_2(x)$; then solving problem (P) amounts to solving problem $(\overline{P})$

$$\min\left\{t\frac{f_3(x)}{f_4(x)} : x \in D, \quad \frac{f_1(x)}{f_2(x)} = t\right\}. \tag{$\overline{P}$}$$

Note that due to the constraint $f_1(x)/f_2(x) = t$, the feasible domain of Problem $(\overline{P})$ may be not convex. Thus $(\overline{P})$ is not of the form of the problems under consideration in [8]. However, the idea of lower bounding operation proposed in [8] could be applied to $(\overline{P})$. Namely, let

$$t_0 := \min_{x \in D} \frac{f_1(x)}{f_2(x)}, \qquad T_0 := \max_{x \in D} \frac{f_1(x)}{f_2(x)},$$

then for every subinterval $I := [t_1, t_2] \subseteq [t_0, T_0]$ we have $\beta(I) \leqslant \alpha(I)$, where

$$\beta(I) := \min_{i=1,2}\left(\min_{x}\left\{t_i\frac{f_3(x)}{f_4(x)} : x \in D, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2\right\}\right)$$
$$\alpha(I) := \min_{(x,t)}\left\{t\frac{f_3(x)}{f_4(x)} : x \in D, \quad \frac{f_1(x)}{f_2(x)} = t, \quad t_1 \leqslant t \leqslant t_2\right\}.$$

To see this, we observe that the set

$$P := \left\{(x,t) \in R^{n+1} : x \in D, \quad \frac{f_1(x)}{f_2(x)} = t, \quad t_1 \leqslant t \leqslant t_2\right\}$$

is contained in the set

$$Q := \left\{(x,t) : x \in D, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2, \quad t_1 \leqslant t \leqslant t_2\right\}.$$

Hence

$$\begin{aligned}\alpha(I) &= \min_{(x,t)}\left\{t\frac{f_3(x)}{f_4(x)} : (x,t) \in P\right\} \geqslant \min_{(x,t)}\left\{t\frac{f_3(x)}{f_4(x)} : (x,t) \in Q\right\}\\ &= \min_{x}\min_{i=1,2}\left\{t_i\frac{f_3(x)}{f_4(x)} : x \in D, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2\right\}\\ &= \min_{i=1,2}\min_{x}\left\{t_i\frac{f_3(x)}{f_4(x)} : x \in D, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2\right\} = \beta(I).\end{aligned}$$

Thus to calculate $\beta(I)$ requires minimizing the two affine fractional functions

$$t_i f_3(x)/f_4(x) \quad (i = 1,2)$$

subject to

$$x \in D, \quad t_1 \leqslant f_1(x)/f_2(x) \leqslant t_2.$$

These problems can be solved very efficiently by a variant of the simplex method [1] if D is a polyhedral convex set defined by a system of linear inequalities.

Based on this lower bounding operation we obtain a branch and bound algorithm which can be described as follows:

ALGORITHM 1

Initialization. Compute

$$\begin{aligned}t_0 &:= \min\{f_1(x)/f_2(x) : x \in D\},\\ T_0 &:= \max\{f_1(x)/f_2(x) : x \in D\}.\end{aligned}$$

Let $I_0 = [t_0, T_0]$ and compute

$$\beta(t_0) := \min\left\{t_0\frac{f_3(x)}{f_4(x)} : x \in D\right\}, \quad \beta(T_0) := \min\left\{T_0\frac{f_3(x)}{f_4(x)} : x \in D\right\}.$$

Let u^0, v^0 be the obtained solutions of these programs. Take

$$\beta_0 := \beta(I_0) = \min\{\beta(t_0), \beta(T_0)\}, \quad \alpha_0 := \min\{f(u^0), f(v^0)\}$$

and $x^0 \in \{u^0, v^0\}$ such that $f(x^0) = \alpha_0$. Set $\mathcal{M}_0 := \{I_0\}$, $k \leftarrow 0$ and go to iteration k.

Iteration k $(k = 0, 1, \ldots)$. At the beginning of iteration k we have a collection $\mathcal{M}_k$ of subintervals $I \subseteq I_0$. For each $I \in \mathcal{M}_k$ we have determined $\beta(I)$. Furthermore we have a lower bound β_k and an upper bound α_k for the optimal value f_*, and a feasible point x^k such that $f(x^k) = \alpha_k$.

(a) If $\alpha_k \leqslant \beta_k$, then terminate: x^k is an optimal solution.

(b) If $\alpha_k > \beta_k$, then define

$$\begin{aligned} I_k^- &:= \{\xi \in I_k : \xi \leqslant (\xi_k + f_1(w^k)/f_2(w^k))/2\}, \\ I_k^+ &:= \{\xi \in I_k : \xi \geqslant (\xi_k + f_1(w^k)/f_2(w^k))/2\}, \end{aligned}$$

where ξ_k and w^k correspond to $\beta_k = \beta(I_k)$, i.e.,

$$\beta_k = \xi_k \frac{f_3(w^k)}{f_4(w^k)} = \min\left\{\xi_k \frac{f_3(x)}{f_4(x)} : x \in D, \quad \frac{f_1(x)}{f_2(x)} \in I_k\right\}$$

with ξ_k being an endpoint (vertex) of the interval I_k.

(c) Compute $\beta(I_k^-)$ and $\beta(I_k^+)$. Let z^{k+1} be the best feasible point obtained during the computation of $\beta(I_k^-)$ and $\beta(I_k^+)$. Let

$$\alpha_{k+1} := \min\{\alpha_k, f(z^{k+1})\}, \quad \mathcal{R}_k := \mathcal{M}\backslash I_k \cup \{I_k^-, I_k^+\},$$

x^{k+1} is feasible point such that $\alpha_{k+1} = f(x^{k+1})$,

$$\mathcal{M}_{k+1} := \{I \in \mathcal{R}_k : \beta(I) \leqslant \alpha_{k+1}\}.$$

(d) Select $I_{k+1} \in \mathcal{M}_{k+1}$ such that

$$\beta(I_{k+1}) := \min\{\beta(I) : I \in \mathcal{M}_{k+1}\}.$$

Let $\beta_{k+1} := \beta(I_{k+1})$, increase k by 1 and go to iteration k.

This completes the description of the algorithm.

REMARKS. (1) This algorithm is similar to the one that Muu and Tam used in their earlier paper [9] for minimizing the sum of a convex and the product of two affine functions over a convex set.

(2) For each interval $I = [t_1, t_2] \subset [t_0, T_0]$, to compute $\beta(I)$ we have to solve the following two programs:

$$\min\left\{t_i \frac{f_3(x)}{f_4(x)} : x \in D, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2\right\} (i = 1, 2).$$

Note that these programs differ from each other only by the multiplier in the objective functions. Thus their solutions coincide if $t_1 t_2 \geqslant 0$.

(3) From the definition of w^k we have $f_1(w^k)/f_2(w^k) \in I_k$. Hence both I_k^- and I_k^+ are nonempty. In fact

$$\min\{\xi_k, f_1(w^k)/f_2(w^k)\} \in I_k^-$$

and

$$\max\{\xi_k, f_1(w^k)/f_2(w^k)\} \in I_k^+.$$

Note that if $\xi_k = f_1(w^k)/f_2(w^k)$, then

$$\beta_k = \frac{f_1(w^k)}{f_2(w^k)}\frac{f_3(w^k)}{f_4(w^k)}.$$

This and $\beta_k \leqslant f_*$, $w^k \in D$ imply that w^k is a solution of the problem.

(4) If we are interested only in an ε-solution, then an interval I can be deleted from further consideration if

$$|f_1(w^I)/f_2(w^I) - t^I| \leqslant \varepsilon |f_4(w^I)/f_3(w^I)| \tag{1}$$

where

$$\beta(I) := t^I \frac{f_3(w^I)}{f_4(w^I)} = \min\left\{ t^I \frac{f_3(w^I)}{f_4(w^I)} : x \in D, \frac{f_1(x)}{f_2(x)} \in I \right\},$$

and t^I is an endpoint of the interval I.

Indeed, since $\beta(I) := t^I f_3(w^I)/f_4(w^I)$ is a lower bound of the objective function on the set $x \in D$, $f_1(x)/f_2(x) \in I$ we obtain, from $w^I \in D$, $f_1(w^I)/f_2(w^I) \in I$, that

$$\begin{aligned} f_* - \beta(I) \leqslant f(w^I) - \beta(I) &= \frac{f_1(w^I)}{f_2(w^I)}\frac{f_3(w^I)}{f_4(w^I)} - t^I \frac{f_3(w^I)}{f_4(w^I)} \\ &= \frac{f_3(w^I)}{f_4(w^I)}\left(\frac{f_1(w^I)}{f_2(w^I)} - t^I\right) \leqslant \varepsilon. \end{aligned}$$

In particular, if (1) holds for I_k, then w^k is an ε-solution to (P).

(5) The above algorithm can be applied to the following problem:

$$\min\left\{ \frac{f_1(x)}{f_2(x)}\frac{f_3(x)}{f_4(x)} + \frac{f_5(x)}{f_6(x)} : x \in D \right\},$$

where f_i $(i = 1, \ldots, 6)$ are affine functions defined by $f_i(x) = c_i^T x + b_i$ with $c_i \in R^n$, $b_i \in R$.

In this case, for each interval $I = [t_1, t_2]$ the lower bound $\beta(I)$ can be calculated similarly as before. Namely, let

$$\beta(t_i) = \min\left\{t_i\frac{f_3(x)}{f_4(x)} + \frac{f_5(2)}{f_6(x)} : x \in D, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2\right\} (i = 1, 2).$$

Then $\beta(I) = \min_{i=1,2} \beta(t_i)$ is a lower bound for

$$\alpha(I) := \min\left\{f(x) : x \in D, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2\right\}.$$

Thus to calculate $\beta(I)$ we have to minimize the sum of two affine fractional functions over a convex set. If D is a polyhedron defined by $D := \{x \in R^n : Ax = b, x \geqslant 0\}$, then this program can be solved efficiently by the algorithm proposed in [6].

Now we turn to the convergence of the algorithm.

THEOREM 1. *(i) If the algorithm terminates at some iteration k, then $\alpha_k = \beta_k = f_*$, and x^k solves (P).*

(ii) If the algorithm is infinite, then $\beta_k \nearrow f_*$, $\alpha_k \searrow f_*$, and any limit point of $\{x^k\}$ solves (P).

Proof. (i) If the algorithm terminates at some iteration k, then $\alpha_k \leqslant \beta_k$. This and $\beta_k \leqslant f_* \leqslant \alpha_k = f(x^k)$, $x^k \in D$ imply that $\beta_k = f_* = \alpha_k$. Hence (i).

(ii) If the algorithm is infinite, then it generates a nested subsequence of the sequence $\{I_k\}$ of subintervals of I_0 which for the sake of simplicity we also denote by $\{I_k\}$.

Since $I_{k+1} \subset I_k^- \cup I_k^+$ for every k, we have from the branching operation that $I_{k+1} \subset I_k^-$ or $I_{k+1} \subset I_k^+$ for all k.

Consider first the case when $I_{k+1} \subset I_k^-$. Let $\xi'_k := f_1(w^k)/f_2(w^k)$ and $d_k := (\xi_k + \xi'_k)/2$. Since

$$\alpha_k \leqslant \frac{f_1(w^k)}{f_2(w^k)}\frac{f_3(w^k)}{f_4(w^k)}, \quad \beta_k = \xi_k\frac{f_3(w^k)}{f_4(w^k)},$$

we have

$$0 < \alpha_k - \beta_k \leqslant \frac{f_1(w^k)}{f_2(w^k)}\frac{f_3(w^k)}{f_4(w^k)} - \xi_k\frac{f_3(w^k)}{f_4(w^k)} = \frac{f_3(w^k)}{f_4(w^k)}(\xi'_k - \xi_k).$$

Thus if $\xi'_k > \xi_k$, then we use $\xi'_{k+1} \leqslant d_k$ to obtain

$$0 < \alpha_k - \beta_k \leqslant 2\frac{f_3(w^k)}{f_4(w^k)}(\xi'_k - d_k) \leqslant 2\frac{f_3(w^k)}{f_4(w^k)}(\xi'_k - \xi'_{k+1}). \tag{2}$$

If $\xi'_k < \xi_k$, then we use $\xi_{k+1} \leqslant d_k$ to obtain

$$0 < \alpha_k - \beta_k \leqslant \frac{f_3(w^k)}{f_4(w^k)}(\xi_k - d_k) \leqslant \frac{f_3(w^k)}{f_4(w^k)}(\xi_k - \xi_{k+1}). \tag{3}$$

Since both $\{\xi_k\}$, $\{\xi'_k\}$ are contained in I_0, we may assume, taking subsequences if necessary, that $\{\xi_k\}$ and $\{\xi'_k\}$ are convergent. This and boundedness of $f_3(x)/f_4(x)$ on D imply, from (2) and (3), that $\alpha_k - \beta_k \to 0$ as $k \to \infty$.

Assume now that $I_{k+1} \subset I_k^+$. By a similar argument we obtain (2) and (3), and hence $\alpha_k - \beta_k \to 0$ as $k \to \infty$. From this and monotonicity of $\{\beta_k\}$ and $\{\alpha_k\}$ it follows that $\beta_k \nearrow f_*$, $\alpha_k \searrow f_*$ which together with $\alpha_k = f(x^k)$ imply that any limit point of $\{x^k\}$ is a global optimal solution of Problem (P).

3. Linear Fractional Programs with an Additional Constraint Defined by the Product of Two Affine Fractional Functions

In this section we modify the method described in the previous section for solving the following problem, denoted by (P_1):

$$\min\left\{f(x) := \frac{f_5(x)}{f_6(x)} : x \in D, \quad \frac{f_1(x)}{f_2(x)}\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0\right\}, \tag{P_1}$$

where as before f_i $(i = 1, \dots, 6)$ are affine functions defined on the convex set $D \subset R^n$. We assume that f_i $(i = 2, 4, 6)$ are positive on D, $\sigma \in R$.

Constraints defined by the product of two affine fractional functions arise in some optimization models in economics (see, e.g. [12]), where the ratio of two profit rates are required to be greater than a constant. The profit rate is usually assumed to be the ratio of two linear functions, where the numerator is the profit and the denominator is the capital. For example, in a domestic and foreign investment model, let $r_1(x) := f_1(x)/f_4(x)$ and $r_2(x) := f_2(x)/f_3(x)$ denote the profit rates of domestic and foreign investment respectively. Usually besides the linear constraints $r_i(x) \geqslant \sigma_i$ $(i = 1, 2)$ the constraint $r_1(x)/r_2(x) \geqslant \alpha$ is introduced. With $\alpha > 1$ this constraint means that the foreign investment is less desirable than the domestic one. Convex programming problems with an additional constraint defined by the product of two convex functions is recently considered in [7].

Let us return to Problem (P_1). If we take $t = f_1(x)/f_2(x)$, then (P_1) is reduced to the following problem, denoted by $(\overline{P_1})$:

$$\min_{(x,t)}\left\{\frac{f_5(x)}{f_6(x)} : x \in D, \quad t\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0, \quad t = \frac{f_1(x)}{f_2(x)}\right\}. \tag{$\overline{P_1}$}$$

Let

$$t_0 := \min_{x \in D} \frac{f_1(x)}{f_2(x)}, \qquad T_0 := \max_{x \in D} \frac{f_1(x)}{f_2(x)}.$$

For each subinterval $I = [t_1, t_2] \subseteq I_0 = [t_0, T_0]$, let

$$\beta(t_i) = \min_x\left\{\frac{f_5(x)}{f_6(x)} : x \in D, \quad t_i\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2\right\}$$
$$(i = 1, 2)$$

and

$$\beta(I) = \min_{i=1,2} \beta(t_i)$$

Then

$$\beta(I) \leqslant \alpha(I) = \min_{(x,t)} \left\{ \frac{f_5(x)}{f_6(x)} : x \in D,\ t\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0,\ t \in I,\ t = \frac{f_1(x)}{f_2(x)} \right\}.$$

This can be seen by an argument similar to the previous one. Let $\xi_i \in \{t_1, t_2\}$ and x^I correspond to $\beta(I)$, i.e.,

$$\begin{aligned} \beta(I) &= \frac{f_5(x^I)}{f_6(x^I)} \\ &= \min\left\{ \frac{f_5(x)}{f_6(x)}, x \in D,\quad \xi_i\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0, t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2 \right\}. \end{aligned}$$

It is clear that if $\xi_i = f_1(x^I)/f_2(x^I)$, then x^I is a feasible point of (P_1). In this case the interval I may be deleted from further consideration. It may occur that during the bounding operation no feasible point of (P_1) is obtained, since the feasible domain of (P_1) may be not convex or even not connected. Thus we need to modify Algorithm 1 for solving Problem (P_1). As usually, we adopt the convention that the minimum of a function over an empty set equals $+\infty$.

ALGORITHM 2 *Initialization.* Compute

$$t_0 := \min\left\{ \frac{f_1(x)}{f_2(x)} : x \in D \right\}, \qquad T_0 := \max\left\{ \frac{f_1(x)}{f_2(x)} : x \in D \right\}.$$

Set $I_0 := [t_0, T_0], \Gamma_0 := \{I_0\}, \mathcal{N}_0 = \Gamma_0$.

Let α_0 be the best currently known upper bound of f_* (if such a bound is not available, let $\alpha_0 = +\infty$).

Iteration k $(k = 0, 1, \ldots)$.

(1) For each $I = [t_1, t_2] \in \mathcal{N}_k$ we compute $\beta(I)$ by solving two linear fractional programs:

$$\min\left\{ \frac{f_5(x)}{f_6(x)} : x \in D,\quad t_i\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0,\quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2 \right\} (i = 1, 2).$$

If feasible points are obtained, update the upper bound α_k and the best currently known feasible point x^k such that $f(x^k) = \alpha_k$. If no feasible point is available, let $\alpha_k = +\infty$.

(2) Delete all $I \in \Gamma_k$ such that $\beta(I) \geqslant \alpha_k$. Let $\mathcal{R}_k$ be the collection of remaining intervals.

(a) If $\mathcal{R}_k = \emptyset$, then terminate: x^k is an optimal solution $(\alpha_k < \infty)$ or (P_1) is infeasible $(\alpha_k = +\infty)$.

(b) If $\mathcal{R}_k \neq \emptyset$, then select $I_k \in \mathcal{R}_k$ such that

$$\beta_k := \beta(I_k) = \min\{\beta(I) : I \in \mathcal{R}_k\}.$$

Let $\xi_k \in I_k$ and $w^k \in D$ corresponding to β_k, i.e.,

$$\beta_k = f(w^k) = \min\left\{f(x) : x \in D, \quad \xi_k \frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0, \quad \frac{f_1(x)}{f_2(x)} \in I_k\right\}.$$

Bisect I_k into the two intervals I_k^- and I_k^+ as follows:

$$I_k^- := \left\{t \in I_k : t \leqslant \left(\xi_k + \frac{f_1(w^k)}{f_2(w^k)}\right)/2\right\},$$
$$I_k^+ := \left\{t \in I_k : t \geqslant \left(\xi_k + \frac{f_1(w^k)}{f_2(w^k)}\right)/2\right\}.$$

(3) Set $\Gamma_{k+1} := \Gamma_k \backslash \{I_k\} \cup \{I_k^-, I_k^+\}$, $\mathcal{N}_{k+1} := \{I_k^-, I_k^+\}$.
Increase k by 1 and go to iteration k.

REMARKS. (1) If $f_3(x) \geqslant 0$ for every $x \in D$ such that $t_1 \leqslant f_1(x)/f_2(x) \leqslant t_2$, then

$$\beta(I) = \min\left\{\frac{f_5(x)}{f_6(x)} : x \in D, \quad t_1 \frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0, \quad t_1 \leqslant \frac{f_1(2)}{f_2(x)} \leqslant t_2\right\}.$$

Likewise, if $f_3(x) < 0$ for every $x \in D, t_1 \leqslant f_1(x)/f_2(x) \leqslant t_2$, then

$$\beta(I) = \min\left\{\frac{f_5(x)}{f_6(x)} : x \in D, \quad t_1 \frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0, \quad t_1 \leqslant \frac{f_1(x)}{f_2(x)} \leqslant t_2\right\}.$$

Thus in these cases we have to solve only one linear fractional program for computing $\beta(I)$,

(2) If $|\xi_k - f_1(w^k)/f_2(w^k)| \leqslant \varepsilon . f_4(w^k)/|f_3(w^k)|$, we may consider w^k as an ε-solution in the sense that $f(w^k) \leqslant f_*$, $w^k \in D$ and w^k is ε-feasible for the constraint

$$\frac{f_1(x)}{f_2(x)}\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0, \quad \text{i.e.} \quad \frac{f_1(w^k)}{f_2(w^k)}\frac{f_3(w^k)}{f_4(w^k)} + \sigma \leqslant \varepsilon.$$

Indeed, since

$$\xi_k \frac{f_3(w^k)}{f_4(w^k)} + \sigma < 0$$

we have

$$\frac{f_1(w^k)}{f_2(w^k)}\frac{f_3(w^k)}{f_4(w^k)} + \sigma \leqslant \frac{f_1(w^k)}{f_2(w^k)}\frac{f_3(w^k)}{f_4(w^k)} + \sigma - \xi_k \frac{f_3(w^k)}{f_4(w^k)} - \sigma$$
$$= \frac{f_3(w^k)}{f_4(w^k)}\left(\frac{f_1(w^k)}{f_2(w^k)} - \xi_k\right) \leqslant \frac{|f_3(w^k)|}{f_4(w^k)}\left|\frac{f_1(w^k)}{f_2(w^k)} - \xi_k\right| \leqslant \varepsilon.$$

(3) Algorithm 2 can be used for solving Problem (P_1) where the objective function is a continuous convex function. In this case the subprograms encountered in determining the lower bounds are convex programming problems.

The following convergence theorem can be proved by an analogous argument as the one in the proof of Theorem 1.

THEOREM 2. *(i) If the algorithm terminates at some iteration k, then $\alpha_k = \beta_k$ is the optimal value of Problem (P_1).*

(ii) If the algorithm is infinite, then $\beta_k \nearrow f_*$, and the sequence $\{w^k\}$ has a limit point which solves (P_1).

(iii) If during the execution of the algorithm a feasible point is available, then $\alpha_k \searrow f_*$ and any limit point of the sequence $\{x^k\}$ solves (P_1).

REMARKS. (1) If in Algorithm 2, at each iteration k we choose $\overline{w}^k$ such that

$$\overline{w}^k \in \arg\min_j \left\{ \frac{f_1(w^j)}{f_2(w^j)}\frac{f_3(w^j)}{f_4(w^j)} : 1 \leqslant j \leqslant k \right\},$$

then the sequence

$$\left\{ \frac{f_1(\overline{w}^k)}{f_2(\overline{w}^k)}\frac{f_3(\overline{w}^k)}{f_4(\overline{w}^k)} \right\}$$

is nonincreasing. Thus, from *(ii)* it follows that any limit point of $\{\overline{w}^k\}$ is an optimal solution to (P_1).

(2) In the two algorithms presented above one can use midpoint bisection [3] i.e., if $I_k = [t_1, t_2]$, then

$$I_k^- := \{t \in I_k, \quad t \leqslant (t_1 + t_2)/2\},$$
$$I_k^+ := \{t \in I_k, \quad t \geqslant (t_1 + t_2)/2\}.$$

This branching operation does not take into account the information obtained from the bounding operation as well as the functions involved in the problems. However, the convergence of the algorithms with this branching operation is also ensured due to the fact that any infinite nested sequence of intervals generated by it tends to a single point [3].

(3) From the above algorithms it is easy to see that they can be used for solving Problems (P) and (P_1) where the product of two affine fractional functions is replaced by the function

$$F(x) := l\left(\frac{f_1(x)}{f_2(x)}\right)\frac{f_3(x)}{f_4(x)}$$

with l being a continuous function of one variable whose global minimum and maximum over a compact interval can be found easily. Example for such a function is any concave or convex function. In this case the lower bound $\beta(I)$ is given by

$$\beta(I) = \min\{\beta_*(I), \beta^*(I)\}$$

where, for Problem (P),

$$\begin{aligned}\beta_*(I) &:= \min\left\{l_*\frac{f_3(x)}{f_4(x)} : x \in D,\quad \frac{f_1(x)}{f_2(x)} \in I\right\},\\ \beta^*(I) &:= \min\left\{-l^*\frac{f_3(x)}{f_4(x)} : x \in D,\quad \frac{f_1(x)}{f_2(x)} \in I\right\},\end{aligned}$$

and for Problem (P_1)

$$\begin{aligned}\beta_*(I) &:= \min\left\{\frac{f_5(x)}{f_6(x)} : x \in D,\quad l_*\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0,\quad \frac{f_3(x)}{f_4(x)} \in I\right\},\\ \beta^*(I) &:= \min\left\{\frac{f_5(x)}{f_6(x)} : x \in D,\quad -l_*\frac{f_3(x)}{f_4(x)} + \sigma \leqslant 0,\quad \frac{f_3(x)}{f_4(x)} \in I\right\},\end{aligned}$$

with

$$l_* := \min\{l(t) : t \in I\},\quad l^* := -\max\{l(t) : t \in I\}$$

for both cases.

Finally, it follows, from the rule for calculating $\beta(I)$, that if the function $[f_3(x)/f_4(x)] > 0$ for every $x \in D$, then we do not require that a global maximum of l over a compact interval is easy to find. Likewise, if $[f_3(x)/f_4(x)] < 0$ on D, then the assumption that the global minimum of l over a compact interval is easy to find may be avoided.

4. Computational Experiments

We now present some preliminary results from computational experiments of the algorithms proposed in Section 2 and Section 3. We solved Problem (P) and (P_1) with a polyhedral convex set D given by

$$D := \{Ax \leqslant b,\ x \geqslant 0\},$$

where $x \in R^n$, A is an $m \times n$ matrix and $b \in R^m$. All elements of A, b and $c_i \in R^n$ $(i = 1, \ldots, 6)$ were randomly generated together with their signs. The program was coded in FORTRAN 77 and tested on a computer IBM/PC AT Turbo 286.

TABLE I.

Prob	M	N	NLFP	MAXINT	ITER	TIME	Algorit
1	10	50	77	21	54	14.43	1
2	10	80	58	14	35	14.93	1
3	10	100	65	17	42	24.06	1
4	15	40	127	30	83	52.47	1
5	15	60	43	6	23	32.67	1
6	15	100	69	15	42	70.96	1
7	20	100	112	29	69	217.87	1
8	10	15	54	4	15	5.6	2
9	10	80	34	1	9	15.4	2
10	15	40	34	1	8	19.4	2
11	15	60	26	1	6	21.9	2
12	20	30	118	12	31	107.3	2
13	20	40	14	1	3	18.1	2

N: number of variables
M: number of constraints (without constraints $x \geqslant 0$)
NLFP: number of linear fractional programs
MAXINT: maximum number of the intervals stored in the memory
ITER: number of iterations
TIME: CPU time (in seconds).
In the tested problems we terminate the program if $|\xi_k - f_1(w^k)/f_2(w^k)| \leqslant \varepsilon|f(w^k)|$, where $\varepsilon = 10^{-5}$.

The results of the computational experiments are shown in Table I.

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